Inverse and determinant of partitioned symmetric matrix

Theorem 1

$$(A+CBD)^{-1} = A^{-1} - A^{-1}C(B^{-1}+DA^{-1}C)^{-1}DA^{-1}$$

Proof:

$$\begin{aligned}
& (A+CBD)[A^{-1} - A^{-1}C(B^{-1}+DA^{-1}C)^{-1}DA^{-1}] \\
= & (A+CBD)A^{-1} - (A+CBD)A^{-1}C(B^{-1}+DA^{-1}C)^{-1}DA^{-1} \\
= & I + CBDA^{-1} - (C+CBDA^{-1}C)(B^{-1}+DA^{-1}C)^{-1}DA^{-1} \\
= & I + CBDA^{-1} - CB(B^{-1}+DA^{-1}C)(B^{-1}+DA^{-1}C)^{-1}DA^{-1} \\
= & I + CBDA^{-1} - CBDA^{-1} = I
\end{aligned}$$

Theorem 2 (inverse of a partitioned symmetric matrix)

Divide an $n \times n$ symmetric matrix A into four blocks

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$$

The inverse matrix $B = A^{-1}$ can also be divided into four blocks:

$$B = A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix}$$

Here we assume the dimensionalities of these blocks are:

- A_{11} and B_{11} are $p \times p$,
- A_{22} and B_{22} are $q \times q$,
- $A_{12} = A_{21}^T$ and $B_{12} = B_{21}^T$ are $p \times q$

with $p+q=n$. Then we have

$$\begin{aligned}
B_{11} &= (A_{11}-A_{12}A_{22}^{-1}A_{12}^T)^{-1} = A_{11}^{-1}+A_{11}^{-1}A_{12}(A_{22}-A_{12}^TA_{11}^{-1}A_{12})^{-1}A_{12}^TA_{11}^{-1} \\
B_{22} &= (A_{22}-A_{12}^TA_{11}^{-1}A_{12})^{-1} = A_{22}^{-1}+A_{22}^{-1}A_{12}^T(A_{11}-A_{12}A_{22}^{-1}A_{12}^T)^{-1}A_{12}A_{22}^{-1} \\
B_{12}^T &= -A_{22}^{-1}A_{12}^T(A_{11}-A_{12}A_{22}^{-1}A_{12}^T)^{-1} \\
B_{12}^T &= -A_{11}^{-1}A_{12}(A_{22}-A_{12}^TA_{11}^{-1}A_{12})^{-1}
\end{aligned}$$

Proof:

$$\begin{aligned}
I_n &= AA^{-1} = AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix} \\
&= \begin{bmatrix} A_{11}B_{11}+A_{12}B_{12}^T & A_{11}B_{12}+A_{12}B_{22} \\ A_{12}^TB_{11}+A_{22}B_{12}^T & A_{12}^TB_{12}+A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}
\end{aligned}$$

i.e.,

$$A_{11}B_{11}+A_{12}B_{12}^T = I_p \quad \text{or} \quad B_{11} = A_{11}^{-1}-A_{11}^{-1}A_{12}B_{12}^T$$

$$A_{11}B_{12}+A_{12}B_{22} = 0 \quad \text{or} \quad B_{12} = -A_{11}^{-1}A_{12}B_{22}$$

$$A_{12}^TB_{11}+A_{22}B_{12}^T = 0 \quad \text{or} \quad B_{12}^T = -A_{22}^{-1}A_{12}^TB_{11}$$

$$A_{12}^TB_{12}+A_{22}B_{22} = I_q \quad \text{or} \quad B_{22} = A_{22}^{-1}-A_{22}^{-1}A_{12}^TB_{12}$$

Plug B_{12}^T into B_{11} to get

$$B_{11} = A_{11}^{-1}+A_{11}^{-1}A_{12}A_{22}^{-1}A_{12}^TB_{11}$$

or

$$(I - A_{11}^{-1}A_{12}A_{22}^{-1}A_{12}^T)B_{11} = A_{11}^{-1} \quad \text{or} \quad (A_{11} - A_{12}A_{22}^{-1}A_{12}^T)B_{11} = I_p$$

or

$$B_{11} = (A_{11} - A_{12}A_{22}^{-1}A_{12}^T)^{-1}$$

Applying theorem 1 to this expression, we also get the other expression in the theorem. Similarly we can get

$$B_{22} = (A_{22} - A_{12}^TA_{11}^{-1}A_{12})^{-1}$$

and

$$B_{12}^T = -A_{22}^{-1}A_{12}^T(A_{11} - A_{12}A_{22}^{-1}A_{12}^T)^{-1}$$

$$B_{12} = -A_{11}^{-1}A_{12}(A_{22} - A_{12}^TA_{11}^{-1}A_{12}^{-1}$$

Theorem 3 (Determinant of a partitioned symmetric matrix)

$$|A| = \left|\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right| = |A_{22}||A_{11} - A_{12}A_{22}^{-1}A_{12}^T| = |A_{11}||A_{22} - A_{12}^TA_{11}^{-1}A_{12}|$$

Proof: Note that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{12}^T & I \end{bmatrix}\begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & A_{22} - A_{12}^TA_{11}^{-1}A_{12} \end{bmatrix}$$

$$= \begin{bmatrix} I & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{12}^T & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix}$$

The theorem is proved as we also know that

$$|AB| = |A|\,|B|$$

and

$$\begin{vmatrix} B & \mathbf{0} \\ C & D \end{vmatrix} = \begin{vmatrix} B & C \\ \mathbf{0} & D \end{vmatrix} = |B| \; |D|$$